

The newest experimental data for the quarks mixing matrix are in better agreement with the *spin-charge-family* theory predictions than the old ones.

G. Bregar, N.S. Mankoč Borštnik

Department of Physics, FMF, University of Ljubljana,
Jadranska 19, SI-1000 Ljubljana, Slovenia

The *spin-charge-family* theory [1–14] predicts before the electroweak break four - rather than the observed three - massless families of quarks and leptons. The 4×4 mass matrices of all the family members demonstrate in this theory the same symmetry, which is determined by the scalar fields: the two $SU(2)$ triplets (the gauge fields of the family groups) and the three singlets, the gauge fields of the three charges (Q, Q' and Y') distinguishing among family members. All the scalars have, with respect to the weak and the hyper charge, the quantum numbers of the *standard model* scalar Higgs [13]: $\pm \frac{1}{2}$ and $\mp \frac{1}{2}$, respectively. Respecting by the *spin-charge-family* theory proposed symmetry of mass matrices and assuming (due to not yet accurate enough experimental data) that the mass matrices are hermitian and real, we fit the six free parameters of each family member mass matrix to the experimental data of twice three measured masses of quarks and to the measured quark mixing matrix elements, within the experimental accuracy. Since any 3×3 submatrix of the 4×4 unitary matrix determines the whole 4×4 matrix uniquely, we are able to predict the properties of the fourth family members provided that the experimental data are enough accurate, which is not yet the case. We, however, found out that the new experimental data [15] for quarks fit better to the required symmetry of mass matrices than the old data [16] and we predict towards which value will more accurately measured matrix elements move. The present accuracy of the experimental data for leptons does not enable us to make sensible predictions.

I. INTRODUCTION

There are several attempts in the literature to reconstruct mass matrices of quarks and leptons out of the observed masses and mixing matrices in order to learn more about properties of the fermion families [17–28]. The most popular is the $n \times n$ matrix, close to the democratic one, predicting that $(n - 1)$ families must be very light in comparison with the n^{th} one. Most of attempts treat neutrinos differently than the other family members, introducing the Majorana part and the "sea-saw" mechanism. Most often are the number of families taken to be equal to the number of the so far observed families, while symmetries of mass matrices are chosen in several

different ways [29–32]. Also possibilities with four families are discussed [33–35].

In this paper we follow the requirements of the *spin-charge-family* theory [1–11, 13, 14], which predicts four coupled families of quarks and leptons and the mass matrix symmetry, which is the same for all the family members.

The mass matrix of each family member is in the *spin-charge-family* theory determined by the scalar fields, which carry besides by the *standard model* required weak and hyper charges [13] ($\pm\frac{1}{2}$ and $\mp\frac{1}{2}$, respectively) also the additional charges: There are two $SU(2)$ triplets, the gauge fields of the family groups, and three singlets, the gauge fields of the three charges (Q, Q' and Y'), which distinguish among family members. These scalar fields cause, after getting nonzero vacuum expectation values [13], the electroweak break. Assuming that the contributions of all the scalar (and in loop corrections also of other) fields to mass matrices of fermions are real and symmetric, we are left with the following symmetry of mass matrices

$$\mathcal{M}^\alpha = \begin{pmatrix} -a_1 - a & e & d & b \\ e & -a_2 - a & b & d \\ d & b & a_2 - a & e \\ b & d & e & a_1 - a \end{pmatrix}^\alpha, \quad (1)$$

the same for all the family members $\alpha \in \{u, d, \nu, e\}$. In App. A 1 the evaluation of this mass matrix is presented and the symmetry commented. The symmetry of the mass matrix Eq.(1) is kept in all loop corrections. A change of phases of the left handed and the right handed basis - there are $(2n - 1)$ free choices - manifests in a change of phases of mass matrices. We made a choice of the simplest phases.

The differences in the properties of the family members originate in different charges of the family members and correspondingly in the different couplings to the corresponding scalar and gauge fields [7].

We fit (sect. III A) the mass matrix elements of Eq. (1) with 6 free parameters for any family member to the so far measured properties of quarks and leptons within the experimental accuracy. That is: *For a pair of either quarks or leptons, we fit twice 6 free parameters of the two mass matrices to twice three so far measured masses and to the corresponding mixing matrix.*

Since we have the same number of free parameters (6 parameters determine in the *spin-charge-family* theory the mass matrix of any family member after the mass matrices are assumed to be real) as there are measured quantities for either quarks or leptons (two times 3 masses and 6 angles of the orthogonal mixing matrix under the simplification that the mixing matrix is real and

hermitian), we should predict the fourth family masses and the missing mixing matrix elements ($V_{u_id4}, V_{u_4d_i}, i \in (1, 2, 3)$) uniquely, provided that the measured quantities are accurate. The $(n - 1)$ submatrix of any unitary matrix determines the unitary matrix uniquely for $n \geq 4$. The experimental inaccuracy, in particular for leptons but also for the matrix elements of the mixing matrix of quarks, is too large to allow us to estimate the fourth family masses even for quarks better than very roughly, if at all.

Yet it turns out that our fitting twice 6 free parameters to the last quarks experimental data [15] leads to better agreement with the data than when we fit to the older data [16].

We predict correspondingly - taking into account new experimental data and the symmetry of the mass matrices of Eq. (1) - how will the quarks 3×3 (sub)matrix (of the 4×4) mixing matrix change in next more accurate measurements.

In our fitting procedure we take into account also the estimations of the influence of the fourth family masses to the decays of mesons from Refs. [43], making also our own estimations (pretty roughly so far, this work is not presented in this paper) [48].

The fact that the new experimental data for quarks fit the symmetry of the mass matrices better than the old ones might be, together with other predictions of this theory [1, 7, 13, 14], a promising signal that the *spin-charge-family* theory is the right step beyond the *standard model* (although we had to assume in this calculations, due to not yet accurate enough quarks mixing matrix, that the mass matrices (Eq. (1)) are real).

In the *spin-charge-family* theory all the family members, the quarks and the leptons, are treated equivalently. However, the experimental data for leptons are so far too inaccurate to allow us to make any accurate enough predictions.

In Sect. II the variational procedure to fit the free parameters of the mass matrices (Eq. (1)) to the experimental data is discussed. In sect. III the numerical results obtained in the fitting procedure of the free parameters of the mass matrices to two kinds of the experimental data [15, 16] are presented. We present the mass matrices for quarks, their diagonal values and the mixing matrix, and we present how good our variational procedure works. The results are commented and predictions made. Discussions are made in Sect. IV.

In App. A we offer a very brief introduction into the *spin-charge-family* theory, which the reader, accepting the proposed symmetry of mass matrices without being curious about the origin of this symmetry, can skip. The rest of appendices are meant as the pedagogical addition.

II. PROCEDURE USED TO FIT FREE PARAMETERS OF MASS MATRICES TO EXPERIMENTAL DATA

Mass matrices Eq.(1), following from the *spin-charge-family* theory, are not in general real (App. B). We, however, assume in this study, due to the experimental inaccuracy of even the quarks mixing matrix, which does not allow us to extract three complex phases, that the mass matrix of any family member (of u and d quarks and ν and e leptons) is real and correspondingly symmetric. We choose the simplest phases for the basic states, as discussed in App. A 1 [49].

The matrix elements of mass matrices, with the loop corrections in all orders taken into account and manifesting the symmetry of Eq. (1), are in this paper taken as free parameters. Due to this symmetry, required by the family quantum numbers of the scalar fields [13], there are 6 parameters having $(n-1) \cdot (n-2)/2$ complex phases. Assuming - simplifying the calculations in accordance with the experimental inaccuracy, which does not allow us to take into account that mass matrices might be complex and the corresponding mixing matrices unitary - that mass matrices are real and mixing matrices are correspondingly orthogonal, there are 6 free real parameters for the mass matrix of any family member - for u and d quarks and for ν and e leptons.

Let us first briefly overview properties of mixing matrices, a more detailed explanation of which can be found in App. A 1.

Let M^α , α denotes the family member ($\alpha = u, d, \nu, e$), be the mass matrix in the massless basis (with all loop corrections taken into account). Let $V_{\alpha\beta} = S^\alpha S^{\beta\dagger}$, where α represents either the u -quark and β the d -quark, or α represents the ν -lepton and β the e -lepton, denotes a (in general unitary) mixing matrix of a particular pair: the quarks one or the leptons one.

For $n \times n$ matrix ($n = 4$ in our case) it follows:

- i. Known matrix elements of the submatrix $(n-1) \times (n-1)$ of an unitary matrix $n \times n$, $n \geq 4$ determine the whole unitary matrix $n \times n$ uniquely: The n^2 unitarity conditions determine $(2(2(n-1)+1))$ real unknowns completely. If the sub matrix $(n-1) \times (n-1)$ of an unitary matrix is made unitary by itself, then we loose the information of the last row and last column.
- ii. If the mixing matrix is assumed to be orthogonal, then the $(n-1) \times (n-1)$ submatrix contains all the information about the $n \times n$ orthogonal matrix to which it belongs and the $n(n+1)/2$ conditions determine the $2(n-1)+1$ real unknowns completely for any n . If the submatrix of the orthogonal matrix is made orthogonal by itself, then we loose all the information of the last row and last column.

In what follows we present the procedure used in our study and repeat the assumptions.

1. If the mass matrix M^α is hermitian, then the unitary matrices S^α and T^α , introduced in appendix B to diagonalize a non hermitian mass matrix, differ only in phase factors depending on phases of basic vectors and manifesting in two diagonal matrices, $F^{\alpha S}$ and $F^{\alpha T}$, corresponding to the left handed and the right handed basis, respectively. For hermitian mass matrices we therefore have: $T^\alpha = S^\alpha F^{\alpha S} F^{\alpha T \dagger}$. By changing phases of basic vectors we can change phases of $(2n - 1)$ matrix elements.
2. We take for quarks and leptons twice three out of twice four diagonal values of the diagonal matrix \mathcal{M}_d^α and the corresponding mixing matrix $V_{\alpha\beta}$ from the available experimental data. The fourth family members properties are to be determined, together with the corresponding mixing matrix elements, from the numerical procedure. Each mass matrix M^α , Eq. (1), has, if it is real, 6 free real parameters $(a^\alpha, a_1^\alpha, a_2^\alpha, b^\alpha, e^\alpha, d^\alpha)$, $\alpha = (u, d, \nu, e)$.
3. We limit the number of free parameters of the mass matrix of each family member α by taking into account n relations among free parameters, in our case $n = 4$, determined by the invariants

$$\begin{aligned}
I_1^\alpha &= - \sum_{i=1,4} m_i^\alpha, & I_2^\alpha &= \sum_{i>j=1,4} m_i^\alpha m_j^\alpha, \\
I_3^\alpha &= - \sum_{i>j>k=1,4} m_i^\alpha m_j^\alpha m_k^\alpha, & I_4^\alpha &= m_1^\alpha m_2^\alpha m_3^\alpha m_4^\alpha, \\
\alpha &= u, d, \nu, e,
\end{aligned} \tag{2}$$

which are the expressions appearing at powers of λ_α , $\lambda_\alpha^4 + \lambda_\alpha^3 I_1 + \lambda_\alpha^2 I_2 + \lambda_\alpha I_3 + \lambda_\alpha^0 I_4 = 0$, in the eigenvalue equation. The invariants are fixed (within the experimental inaccuracy of the data) by the observed masses of quarks and leptons and by the chosen value of the fourth family mass m_4^α . One can express the four invariants with the parameters of the mass matrix (Eq. (1)):

$$\begin{aligned}
a^\alpha &= \frac{I_1^\alpha}{4}, \\
-I_2^\alpha + 6(a^\alpha)^2 &= (a_1^\alpha)^2 + (a_2^\alpha)^2 + 2(b^\alpha)^2 + 2(d^\alpha)^2 + 2(e^\alpha)^2, \\
I_3^\alpha - 2a^\alpha I_2^\alpha + 8(a^\alpha)^3 &= -8d^\alpha e^\alpha b^\alpha, \\
I_4^\alpha - a^\alpha I_3^\alpha + (a^\alpha)^2 I_2^\alpha - 3(a^\alpha)^4 &= ((a_1^\alpha)^2 + (b^\alpha)^2)((b^\alpha)^2 + (a_2^\alpha)^2) - 2a_1^\alpha a_2^\alpha ((e^\alpha)^2 - (d^\alpha)^2) - \\
&\quad 2(b^\alpha)^2 ((e^\alpha)^2 + (d^\alpha)^2) + ((e^\alpha)^2 - (d^\alpha)^2)^2.
\end{aligned} \tag{3}$$

Correspondingly there are $(6 - 4)$ free real parameters left for each mass matrix, after a choice is made for the mass of the fourth family member.

4. The diagonalizing matrices S^α and S^β , each depending on $(6 - 4)$ free real parameters, after the choice of the fourth family masses, are for real and symmetric mass matrices orthogonal. They follow from the procedure

$$\begin{aligned} M^\alpha &= S^\alpha \mathbf{M}_d^\alpha T^{\alpha\dagger}, \quad T^\alpha = S^\alpha F^{\alpha S} F^{\alpha T\dagger}, \\ \mathbf{M}_d^\alpha &= (m_1^\alpha, m_2^\alpha, m_3^\alpha, m_4^\alpha), \end{aligned} \quad (4)$$

provided that S^α and S^β fit the experimentally observed mixing matrices $V_{\alpha\beta}^\dagger$ within the experimental accuracy and that M^α and M^β manifest the symmetry presented in Eq. (1). We keep the symmetry of the mass matrices accurate. One can proceed in two ways.

$$\begin{aligned} A. : \quad S^\beta &= V_{\alpha\beta}^\dagger S^\alpha, & B. : \quad S^\alpha &= V_{\alpha\beta} S^\beta, \\ A. : \quad V_{\alpha\beta}^\dagger S^\alpha \mathbf{M}_d^\beta S^{\alpha\dagger} V_{\alpha\beta} &= M^\beta, & B. : \quad V_{\alpha\beta} S^\beta \mathbf{M}_d^\alpha S^{\beta\dagger} V_{\alpha\beta}^\dagger &= M^\alpha. \end{aligned} \quad (5)$$

The indices α and β determine to which family member the matrix S corresponds, or to which two family members the mixing matrix V corresponds. In the case *A.* one obtains from Eq. (4), after requiring that the mass matrix M^α has the desired symmetry, the matrix S^α and the mass matrix M^α ($= S^\alpha \mathbf{M}_d^\alpha S^{\alpha\dagger}$), from where we get the mass matrix $M^\beta = V_{\alpha\beta}^\dagger S^\alpha \mathbf{M}_d^\beta S^{\alpha\dagger} V_{\alpha\beta}$. In case *B.* one obtains equivalently the matrix S^β , from where we get M^α ($= V_{\alpha\beta} S^\beta \mathbf{M}_d^\alpha S^{\beta\dagger} V_{\alpha\beta}^\dagger$). We use both ways iteratively taking into account the experimental accuracy of masses and mixing matrices.

5. Under the assumption of the present study that the mass matrices are real, form the corresponding orthogonal diagonalizing matrices S^α and S^β the orthogonal mixing matrix $V_{\alpha\beta}$, which depends on at most $6 (= \frac{n(n-1)}{2})$ free real parameters (App. B).

We should not forget, that the assumption of the real and symmetric mass matrices, leading to orthogonal mixing matrices, might not be an acceptable simplification. However, the experimental inaccuracy in particular of the mixing matrices does not allow us (yet) to take complex phases into account. (In the next step of study, with hopefully more accurate experimental data, we will relax conditions on hermiticity of mass matrices and correspondingly on orthogonality of mixing matrices, allowing them to be unitary.) We expect and shall find that too large experimental inaccuracy leaves the fourth family masses in the present study quite undetermined, even for quarks.

6. We study quarks and leptons equivalently.

7. The difference among family members originate in the eigenvalues of the operators $(Q^\alpha, Q'^\alpha, Y'^\alpha)$, which in loop corrections in all orders contribute to all mass matrix elements, causing the difference among family members [50].

Let us summarize. If the mass matrix of a family member obeys the symmetry required by the *spin-charge-family* theory, which in a simplified version is assumed to be real and symmetric (as it is taken in this study), the matrix elements of the mixing matrices of either quarks or leptons are correspondingly real, each of them with $\frac{n(n-1)}{2} = 6$ parameters. The 2×6 free parameters of the two mass matrices (of either the quark or the lepton pair) are in this case determined by 2×3 measured masses and the 6 parameters of the mixing matrix of the corresponding pair. The accurate enough experimental data would therefore determine the fourth family masses of each family member and the mixing matrix elements of the fourth families members for each pair.

Since the so far measured masses and in particular the measured mixing matrices are not determined accurately enough, we can in the best case expect that the masses and the mixing matrix elements of the fourth families will be determined only within some (quite large) intervals.

In the Subsect. II A we present in more details the procedure used to determine free parameters of mass matrices by taking into account the experimental inaccuracy.

A. Free parameters of mass matrices after taking into account invariants

We present in this subsection the numerical procedure used in our calculations for fitting free parameters of each family member mass matrix $(a^\alpha, a_1^\alpha, a_2^\alpha, b^\alpha, e^\alpha, d^\alpha; \alpha = (u, d, \nu, e))$, Eq. (1), to the experimental data: 2×3 masses and the 3×3 submatrix of the corresponding 4×4 unitary mixing matrix for either quarks or leptons.

We make the assumption that the mass matrices (Eq. (1)) are real, since the elements of the mixing matrices are not known accurately enough to extract 3 complex phases of the unitary 4×4 mixing matrix, either for quarks or (even much less) for leptons. Correspondingly are the diagonalizing matrices orthogonal and so are the mixing matrices.

The accurate 3×3 submatrix of the 4×4 unitary matrix would determine the unitary matrix uniquely. The assumption that the mass matrices (Eq. (1)) are real makes the mixing matrix orthogonal, determined by 6 parameters. These 6 parameters and twice 3 masses of the pair of either quarks or leptons, if accurate, would determine twice 6 parameters of mass matrices uniquely. It is the experimental inaccuracy, which makes this fitting procedure very demanding, in particular since the 3×3 submatrix of the 4×4 mixing matrix is, within the experimental inaccuracy, close

to an unitary matrix.

Our variational procedure must find the best fit to the experimental data keeping the symmetry of any of the mass matrices of a pair of quarks or leptons presented in Eq. (1).

The result of the variational procedure are correspondingly 6 free parameters of each of the two mass matrices of a pair, those with the best fit to the experimental data within their accuracy - for either the mixing matrix or for twice the three masses. The fitting procedure offers correspondingly intervals for the fourth family masses as well as for the mixing matrix elements of the fourth family members to the rest three of the family members.

Although we have investigated leptons and quarks, it has turned out that the experimental data for leptons are not yet accurate enough that we would be able to make some valuable predictions for leptons. We therefore concentrate in this paper on quarks.

We skip in this subsection, for the sake of simplicity, the family member index α . It appears useful, in purpose of numerical evaluation, to take into account for each family member its mass matrix invariants, Eq. (2), which are the coefficients of the characteristic polynomials, and to make a choice of the fourth family masses or rather $a = \frac{1}{4} I_1$ (Eq. (7)) instead of the fourth family mass. We also introduce new parameters f, g, q and r

$$a, b, \quad f = d + e, \quad g = d - e, \quad q = \frac{a_1 + a_2}{\sqrt{2}}, \quad r = \frac{a_1 - a_2}{\sqrt{2}}, \quad (6)$$

and express the parameters a, q, r, b, f and g of the mass matrix, Eq. (1), with invariants, Eq. (2),

$$\begin{aligned} a &= \frac{I_1}{4}, \\ I'_2 &= -I_2 + 6a^2 = q^2 + r^2 + 2b^2 + f^2 + g^2, \\ I'_3 &= I_3 - 2aI_2 + 8a^3 = -2b(f^2 - g^2), \\ I'_4 &= I_4 - aI_3 + a^2I_2 - 3a^4 \\ &= \frac{1}{4}(q^2 - r^2)^2 + (q^2 + r^2)b^2 + \frac{1}{2}(q^2 - r^2) \cdot 2gf - b^2(f^2 + g^2) + \frac{1}{4}(2gf)^2 + b^4. \end{aligned} \quad (7)$$

We do this to reduce the number of free parameters for each mass matrix from 6 to 3 by making a choice of (twice) 3 masses within the experimentally allowed values. It turns out, namely, when testing the stability of the variational procedure against changing masses of the family members within the experimental inaccuracy, that the experimental inaccuracy of twice three known masses of a pair of two family members (either quarks or leptons) influences parameters of the mass matrices and correspondingly the fourth family masses and the corresponding mixing matrix much less than the inaccuracy of the matrix elements of the 3×3 mixing submatrix. We test the

stability of the variational procedure against changing masses of the family members within the experimental inaccuracy after each variational procedure.

We eliminate, using the second and the third equation of Eq. (7), the parameters f and g , expressing them as functions of I'_2 and I'_3 and the parameters a, r, q and b

$$\begin{aligned} f^2 &= \frac{1}{2} \left(I'_2 - q^2 - r^2 - 2b^2 - \frac{I'_3}{2b} \right) \\ g^2 &= \frac{1}{2} \left(I'_2 - q^2 - r^2 - 2b^2 + \frac{I'_3}{2b} \right). \end{aligned} \quad (8)$$

To avoid that the product gf appearing in the last equation of Eq. (7) would be expressed by the square root of the right hand sides of Eq. (8) we square the last equation of Eq. (7) (after putting gf on one side of the equation and all the rest on the other side of the equation). The last four free parameters (a, b, q, r) are now related as follows

$$\begin{aligned} &\left\{ -\frac{1}{2}(q^4 + r^4) + (-2b^2 + \frac{1}{2}(I'_2 - 2b^2))(q^2 + r^2) \right. \\ &+ \left. (I'_4 - b^4 - \frac{1}{4}((I'_2 - 2b^2)^2 - I_3'^2/4b^2) + b^2(I'_2 - 2b^2)) \right\}^2 \\ &= \frac{1}{4}(q^2 - r^2)^2((I'_2 - 2b^2 - (q^2 + r^2))^2 - I_3'^2/4b^2), \end{aligned} \quad (9)$$

what reduces the number of free parameters to 3 for each member of the pair, either of quarks or of leptons. Only powers of q^2 contribute, and since the term q^8 cancels we are left with a cubic equation for q^2

$$\beta q^6 + \gamma q^4 + \delta q^2 + \rho = 0. \quad (10)$$

Coefficients $(\beta, \gamma, \delta, \rho)$ depend on the 3 free remaining parameters (a, b, r) and the three (within experimental accuracy) known masses. These twice 3 free parameters for a pair of either quarks or leptons must be determined from the corresponding measured matrix elements of the 3×3 submatrix of the 4×4 mixing matrix (which is assumed to be orthogonal).

We investigate the concordance between the *spin-charge-family*, which defines the symmetry of the mass matrix (Eq. (1)) of each family member, and the experimental data offering masses of twice three families and, due to the theory, the matrix elements of the 3×3 submatrices of the two 4×4 mixing matrices.

We proceed as follows: i. First we make a choice of all the masses of u and d quarks - twice the three known ones within the experimentally allowed values, while the fourth masses are taken as free parameters. ii. Then we look for the best fit of twice six mass matrix parameters - after taking into account all the relations of Eqs. (7, 8, 9, 10) - to the six parameters of the mixing matrix, which

all are determined only within experimental accuracy. We repeat the same procedure with several choices of masses: for the three known masses we chose values within the experimentally allowed intervals, the fourth masses are chosen from 300 GeV to 1200 GeV. The investigations confirm that the experimental uncertainties of the lower three masses of quarks influence our fitting procedure very little.

The fixed masses of all quarks simplify calculations substantially. The parameter $a = -(m_1 + m_2 + m_3 + m_4)/4$ becomes a constant and the only parameters that remain to be determined from the mixing matrix are b and r for the u and the d quarks, that is 4 parameters.

For each type of quarks we look for the allowed region in the parameter space of b and r in which the remaining three parameters (f , g and q) solve the equations (7) as real numbers. This condition requires that f and g , which follow from Eq. (8), are real and that the solution for q from Eq. (9) is also real. We end up with the inequality

$$I'_2 - 2b^2 - (q^2 + r^2) \geq \left| \frac{I'_3}{2b} \right|, \quad (11)$$

which determines the maximal and minimal positive b (both appearing at $q = 0 = r$). Eq. (9) is insensitive to the sign of b . The sign of f and g manifests only in Eq. (8), where the change $b \rightarrow -b$ interchanges f and g ($f \leftrightarrow g$). Since Eq. (9) includes only even powers of q , r and b , the signs need to be studied in a detailed way.

There are several matrix transformations of the kind $M' = P_i M P_i^T$, M is presented in Eq. (1), P_i of which can be read from Eq. (3)

$$\begin{aligned} P_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; & P_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; & P_3 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \\ P_4 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; & P_5 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; & P_6 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \\ P_7 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (12)$$

These transformations cause changes of the parameters of the mass matrix of Eq. (1), and correspondingly also of the new parameters (Eq. (6)) as follows

$$\begin{aligned}
P_1 & : a_1 \rightarrow -a_1; a_2 \rightarrow -a_2; \quad q \rightarrow -q; r \rightarrow -r; \\
P_2 & : a_1 \leftrightarrow a_2; \quad r \rightarrow -r; \\
P_3 & : e \rightarrow -e; b \rightarrow -b; \quad f \leftrightarrow g; \\
P_4 & : d \rightarrow -d; b \rightarrow -b; \quad f \leftrightarrow -g; \\
P_5 & : e \rightarrow -e; d \rightarrow -d; \quad f \rightarrow -f, g \rightarrow -g; \\
P_6 & : e \leftrightarrow d; a_2 \rightarrow -a_2; \quad q \leftrightarrow r; g \rightarrow -g; \\
P_7 & : e \leftrightarrow d; a_1 \rightarrow -a_1; \quad q \leftrightarrow -r; g \rightarrow -g.
\end{aligned} \tag{13}$$

We make use of these transformations in the way that we find the solution of equations (7) for one set of signs of parameters q, r, b, f and g and then obtain all the other sets of solutions using transformations (13).

The procedure regarding the signs is the following: i. We see from Eq. (7) that the signs of q and r can be chosen arbitrarily. ii. Then we chose $b > 0$ and calculate f^2 and g^2 from Eq. (8). iii. We see in Eq. (7) that f and g are present only as powers of f^2, g^2 and fg . To obtain the valid sign of f and g we must therefore determine only the sign of the product fg . iv. The solution for $b < 0$ follows when using the transformation P_4 . v. After obtaining in such a procedure all possibilities, the region of allowed parameters in the space of parameters b and r follows from Eq (11). Since I'_2 is positive by definition, we see that the largest value of positive b is obtained when $q = r = 0$. In this case we have the inequality

$$I'_2 - 2b^2 \geq \left| \frac{I'_3}{2b} \right|, \tag{14}$$

which bounds b from below and from above. The two limits, b_{min} and b_{max} , are the solutions of the equation which follows from the inequality. We introduce a new parameter η_b

$$b = b_{min} + \eta_b(b_{max} - b_{min}). \tag{15}$$

When we fix a certain b with some choice of $\eta_b \in [0, 1]$ then r and q are limited from above, while they are limited from below by 0. Therefore the largest r, r_{max} , solves the equation

$$I'_2 - 2b^2 - r_{max}^2 = \left| \frac{I'_3}{2b} \right|. \tag{16}$$

We introduce a new parameter $\eta_r \in [0, 1]$

$$r = \eta_r r_{max}. \tag{17}$$

For a fixed r , q is limited from above by the inequality

$$I'_2 - 2b^2 - (q_{max}^2 + r^2) \geq \left| \frac{I'_3}{2b} \right|. \quad (18)$$

In the space of new parameters η_b and η_r the cubic equation (9, 10) selects the region in which a regular solution exists. This region depends on the four masses of each family member (m_i ; $i \in (1, \dots, 4)$).

Introducing the polar coordinates, centered at $\eta_b = 1/2$, $\eta_r = 1/2$, one finds that the angle γ lies in the interval $[\gamma_{min}, \gamma_{max}]$, where $\gamma_{min} < 0$. For each chosen γ the radius takes values between $\rho_1(\gamma)$ and $\rho_2(\gamma)$. It is useful to introduce new parameters η_γ and η_ρ as follows

$$\begin{aligned} \gamma &= \gamma_{min} + \eta_\gamma (\gamma_{max} - \gamma_{min}), \\ \rho &= \rho_{min} + \eta_\rho (\rho_{max} - \rho_{min}), \end{aligned} \quad (19)$$

with ρ_{min} and ρ_{max} functionally depending on η_γ .

An example of such a region for the d -quarks with the choice for their masses ($m_1^d = 2.9$ MeV, $m_2^d = 55$ MeV, $m_3^d = 2900$ MeV, $m_4^d = 650\,000$ MeV) is presented on Fig. 1. The allowed region

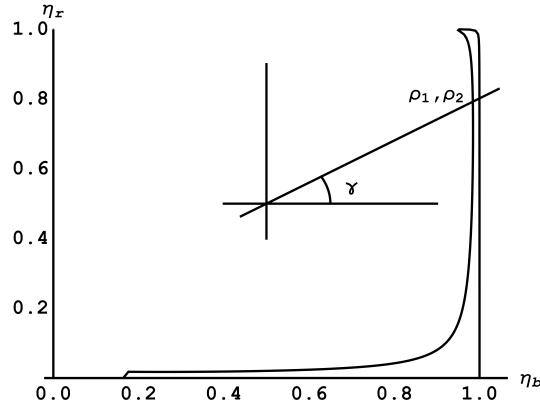


FIG. 1: The allowed region for the d -quarks with masses choice ($m_1^d = 2.9$, $m_2^d = 55$, $m_3^d = 2900$, $m_4^d = 650\,000$) MeV in the space of parameters η_b , η_r .

lies between the curve and the two axes.

The calculation reduces, for a pair of quarks or for a pair of leptons, for the chosen values of twice four masses of a pair (twice three masses taken within by the experimental data determined values, the two fourth ones are taken as parameters) to searching for a minimum of the total discrepancy between the measured mixing matrix elements (with the measuring inaccuracy taken into account) and the calculated elements, which depend on 2×2 parameters, $(\eta_\gamma^u, \eta_\rho^u$ and $\eta_\gamma^d, \eta_\rho^d)$

or $(\eta_\gamma^\nu, \eta_\rho^\nu$ and $\eta_\gamma^e, \eta_\rho^e)$, Eq. (20). We minimize, for each choice of 2×4 of masses of a pair, the uncertainty σ_{ud} (and equivalently for $\sigma_{\nu e}$):

$$\begin{aligned}\sigma_{ud} &= \sqrt{\sum_{(i,j)=1}^3 \left(\frac{V_{u_i d_j \text{ exp}} - V_{u_i d_j \text{ cal}}}{\sigma_{V_{u_i d_j \text{ exp}}}} \right)^2}, \\ \delta V_{u_i d_j} &= \left| \frac{V_{u_i d_j \text{ exp}} - V_{u_i d_j \text{ cal}}}{\sigma_{V_{u_i d_j \text{ exp}}}} \right|,\end{aligned}\tag{20}$$

where the two indexes exp and cal determine experimental and calculated values, respectively, and $\delta V_{u_i d_j}$ determine the discrepancy between the experimental and the calculated values for the 3×3 submatrix of the corresponding 4×4 mixing matrix. The experimental values for the quark mixing matrix, the old ones [16] and the new ones [15], are presented in Eqs. (21, 22), while the used twice three quark masses, recalculated to the energy scale of the mass of Z boson, M_Z , are presented in Eq. (23).

The search in the 4 dimensional unit cube is performed in the way that we fix the two values of parameters η_γ^u and η_γ^d and do a search in the two dimensional plane of the other two parameters.

The minimizing process is repeated for several choices of the experimentally allowed values for twice three measured masses and for several choices of the two fourth family masses in the interval $300 \text{ GeV} \leq (m_{u_4}, m_{d_4}) \leq 1200 \text{ GeV}$ (for leptons we have checked the interval $60 \text{ GeV} \leq (m_{u_4}, m_{d_4}) \leq 1000 \text{ GeV}$).

The numerical procedure, used in this contribution, is designed for quarks and for lepton, although we present in this paper the results for quarks only.

III. NUMERICAL RESULTS

We present in this paper predictions obtained when taking into account by the *spin-charge-family* theory required properties of mass matrices (Eq. (1)) and fitting free parameters of mass matrices to the measured properties of quarks, using the procedure explained in Sects. II, II A. Although we have performed calculations also for leptons, it turned out that the experimental accuracy of the data for leptons are not yet high enough that we could make reliable predictions for them. The experimental accuracy is not high enough even for quarks to make use of one complex phase of the measured matrix elements of the mixing matrix. We therefore make in this paper the assumption that the mass matrices are real. Correspondingly the matrices, which diagonalize mass matrices, are orthogonal and so is orthogonal also the mixing matrix.

The measured 9 matrix elements of the quarks mixing matrix form, within the experimental inaccuracy, almost unitary 3×3 matrix. Correspondingly are the results of our fitting procedure of twice 6 parameters of the two 4×4 mass matrices of Eq. (1) to twice three measured masses and to the measured 9 matrix elements of the 3×3 submatrix of the corresponding 4×4 mixing matrix sensitive mostly to the accuracy of the measured matrix elements of the mixing matrix.

It turns out, as expected, that inaccuracy of the measured mixing matrix elements does not allow us to tell much about the fourth family masses, while inaccuracies of two times three quark masses influence the results much less: Quite large intervals for the fourth family masses change the calculated mixing matrix elements of the 3×3 submatrix very little. The results show that the most trustworthy might be results pushing the fourth family quarks to approximately 1 TeV or above.

What comes out of our calculations is that the new experimental data [15] for the mixing matrix elements, Eq. (22), fit better the predicted symmetry of Eq. (1) than the old ones [16], Eq. (21), and that we *can correspondingly predict, following the spin-charge-family theory, how will the matrix elements of the 3×3 submatrix of the 4×4 mixing matrix change in next measurements.* We predict also the matrix elements of the fourth family members. We shall see that $V_{u_i d_4}$ and $V_{u_4 d_i}$, $i \in (1, 2, 3)$, do not change very so strongly with the increasing fourth family masses, as one would expect. Results of the present paper, together with the results of the previous works [13, 14], support the hope that the *spin-charge-family* theory might be the right next step beyond the *standard model*.

Using the procedure, explained in sect. II (to some extend presented already in Ref. [11]) we are looking for the 4×4 in this paper taken to be real and correspondingly symmetric mass matrices for quarks, obeying the symmetry of Eq. (1), which reproduce in accordance with Eq. (20) as accurately as possible the measured properties - masses and mixing matrices - of the so far observed three families of quarks, and which are in agreement also with the experimental limits for the appearance of the fourth family masses and of the mixing matrix elements to the lower three families, as presented in Refs. [15, 16, 33–35, 43]. (We have made also our own rough estimations for the limitations which follow from the meson decays to which the fourth family members participate. Our estimations are still in progress.)

A lot of effort was put into the numerical procedure (sect. II) to ensure that we fit the parameters of mass matrices to the experimental values within the experimental inaccuracy in the best way, that is with the smallest errors (Eq. (20)).

The results manifest that the mass matrices are close to the democratic ones, which is, as

expected, more and more the case the higher might be the fourth family masses, and it is true for quarks and leptons.

To test the predicting power of our model in dependence of the experimental inaccuracy of masses and mixing matrices, we compare among themselves all the results of the fitting procedure, Eq. (20), obtained when changing the fourth family quark masses in the interval of 300 GeV to 1700 GeV, for either old [16] or new [15] experimental data.

We look for several properties of the obtained mass matrices:

- i. We test the influence of the experimentally declared inaccuracy of the 3×3 submatrices of the 4×4 mixing matrices and of the twice 3 measured masses on the prediction of the fourth family masses.
- ii. We look for how do the old and the new matrix elements of the measured mixing matrix influence the accuracy with which the experimental data are reproduced in the procedure which takes into account the symmetry of mass matrices.
- iii. We look for how different choices for the masses of the fourth family members limit the inaccuracy of particular matrix elements of the mixing matrices or the inaccuracy of the three lower masses of family members.
- iv. We test how close to the democratic mass matrix are the obtained mass matrices in dependence of the fourth family masses.
- v. We look for the predictions of the 4×4 mass matrices with the symmetry presented in Eq. (1).

A. Experimental data used in this calculations

We take for the quark masses the experimental values [16], recalculated to the Z boson mass scale. We take two kinds of the experimental data for the quark mixing matrices with the experimentally declared inaccuracies for the so far measured 3×3 mixing matrix elements: The older data from [16] and the latest data [15]. We assume, as predicted by the *spin-charge-family* theory, that these nine matrix elements belong to the 4×4 unitary mixing matrix.

We first do the calculations, explained in Sect. II, with the older experimental data [16]

$$|V_{ud}| = \begin{pmatrix} 0.97425 \pm 0.00022 & 0.2252 \pm 0.0009 & 0.00415 \pm 0.00049 & |V_{u_1 d_4}| \\ 0.230 \pm 0.011 & 1.006 \pm 0.023 & 0.0409 \pm 0.0011 & |V_{u_2 d_4}| \\ 0.0084 \pm 0.0006 & 0.0429 \pm 0.0026 & 0.89 \pm 0.07 & |V_{u_3 d_4}| \\ |V_{u_4 d_1}| & |V_{u_4 d_2}| & |V_{u_4 d_3}| & |V_{u_4 d_4}| \end{pmatrix}, \quad (21)$$

and then we repeat all the calculations also with the new experimental data [15]

$$|V_{ud}| = \begin{pmatrix} 0.97425 \pm 0.00022 & 0.2253 \pm 0.0008 & 0.00413 \pm 0.00049 & |V_{u_1 d_4}| \\ 0.225 \pm 0.008 & 0.986 \pm 0.016 & 0.0411 \pm 0.0013 & |V_{u_2 d_4}| \\ 0.0084 \pm 0.0006 & 0.0400 \pm 0.0027 & 1.021 \pm 0.032 & |V_{u_3 d_4}| \\ |V_{u_4 d_1}| & |V_{u_4 d_2}| & |V_{u_4 d_3}| & |V_{u_4 d_4}| \end{pmatrix}. \quad (22)$$

For the quark masses at the energy scale of M_Z we take

$$\begin{aligned} \mathbf{M}_d^u/\text{MeV}/c^2 &= (1.3 + 0.50 - 0.42, 619 \pm 84, 172\,000. \pm 760., 300\,000. \leq m_{u_4} \leq 1\,200\,000.), \\ \mathbf{M}_d^d/\text{MeV}/c^2 &= (2.90 + 1.24 - 1.19, 55 + 16 - 15, 2\,900. \pm 90., 300\,000. \leq m_{d_4} \leq 1\,200\,000.). \end{aligned} \quad (23)$$

The matrix elements of the 4×4 quark mixing matrix are determined in the numerical procedure, which searches for the best fit of the two quarks mass matrices free parameters, presented in Eq. (1), to the experimental data, by taking into account the experimental inaccuracy and the unitarity, in this paper orthogonality, of the 4×4 mixing matrix, ensuring as much as possible, the best fit as defined in Eq. (20).

Let us notice that the new experimental data for the quark mixing matrix differ from the old ones the most in the two diagonal matrix elements, $V_{cs} = V_{u_2 d_2}$ and $V_{tb} = V_{u_3 d_3}$, appearing in the new data with smaller inaccuracy. The differences among the old and the new values of $V_{us} = V_{u_1 d_2}$, $V_{ub} = V_{u_1 d_3}$, $V_{cd} = V_{u_2 d_1}$, $V_{cb} = V_{u_2 d_3}$ and $V_{ts} = V_{u_3 d_2}$, are smaller (some of them even with not better accuracy than in the old data) than in the case of the diagonal ones, while the rest two, $V_{ud} = V_{u_1 d_1}$ and $V_{td} = V_{u_3 d_1}$, were not measured more accurately. The values for the quark masses are taken in both cases within the measured inaccuracy.

The corresponding fourth family mixing matrix elements ($|V_{u_i d_4}|$ and $|V_{u_4 d_j}|$) are in both cases determined from the unitarity condition for the 4×4 mixing matrix through the fitting procedure, explained in Sect. II, as also all the other matrix elements of the mixing matrix are.

Using first the old experimental data [16] we predict the direction in which new more accurately measured matrix elements should move and then we check if this is happening with the new experimental data [15].

Then we use the new experimental data, repeat the variational procedure and look for what are our new results predicting.

The results are presented in the next Subsect. III B.

B. The mass matrices for quarks, their masses, the mixing matrix and predictions

As already written, the mixing matrix elements for quarks, forming in the *spin-charge-family* 3×3 submatrix of the 4×4 unitary matrix, are not yet measured accurate enough to allow us a trustworthy prediction of the fourth family quark masses. Correspondingly we perform all the calculation for the chosen set of the fourth family masses and study how does the accuracy of the fitting procedure, explained in Sect. II, depend on the fourth family masses.

Still we can make predictions: **i.** We make a very rough estimation of the fourth family masses. **ii.** We evaluate the matrix elements $V_{u_i d_4}$ and $V_{u_4 d_j}$, $(i, j) \in (1, 2, 3)$ for chosen fourth family quark masses and check their dependence on all the quark masses (with the fourth family included). **iii.** We predict how will the matrix elements $V_{u_i d_j}$ of the 3×3 submatrix of the 4×4 mixing matrix change in next more accurate measurements, under the assumption that the *spin-charge-family* theory is the right next step beyond the *standard model*.

We test the extent to which our results have some experimental support by performing calculations with the two kinds of the experimental values for the quark mixing matrix: The older ones presented in Eq. (21) and the newer ones presented in Eq. (22). We use both data with the same set of the fourth family quark masses.

Below we present results for the two choices of masses: for $m_{u_4} = 700 \text{ GeV} = m_{d_4}$ and for $m_{u_4} = 1200 \text{ GeV} = m_{d_4}$, obtained when fitting twice 6 free parameters of the mass matrices to twice three measured masses of quarks and to their measured mixing matrix, first for the old data [16] and then for the new data [15], taking into account the experimental accuracy, although we test the whole interval for the fourth family masses, from 300 GeV to 1700 GeV. We present the two choices as an illustration.

Looking for the results from the fitting procedure when using the old ([16]) experimental data for the quark mixing matrix, we predict, from the calculated 3×3 submatrix, the expected changes in the new data ([15]) and comment these predictions. Then we repeat calculations with the new data ([15]) and predict, how will the matrix elements change in next more accurate measurements.

We keep the symmetry (Eq. (1)) of mass matrices exact.

- **I.** We present first the results of our calculations when using the old data [16] (Eqs. (23, 21)), and make a choice for the fourth family quark masses: first $m_{u_4} = m_{d_4} = 700 \text{ GeV}$ and then $m_{u_4} = m_{d_4} = 1200 \text{ GeV}$. We present for the best fit (Eq. (20)) to the old experimental data the corresponding mass matrices, their diagonal values, the mixing matrix and the deviations of the calculated matrix elements from the measured ones (Eq. (20)).

1. We choose first $m_{u_4} = 700$ GeV and $m_{d_4} = 700$ GeV and obtain through the variational procedure (Sect. II) the two mass matrices.

$$M^u = \begin{pmatrix} 227623. & 131877. & 132239. & 217653. \\ 131877. & 222116. & 217653. & 132239. \\ 132239. & 217653. & 214195. & 131877. \\ 217653. & 132239. & 131877. & 208687. \end{pmatrix}, M^d = \begin{pmatrix} 175797. & 174263. & 174288. & 175710. \\ 174263. & 175666. & 175710. & 174288. \\ 174288. & 175710. & 175813. & 174263. \\ 175710. & 174288. & 174263. & 175682. \end{pmatrix}, \quad (24)$$

which define the mixing matrix

$$V_{ud} = \begin{pmatrix} -0.97423 & 0.22531 & -0.00299 & 0.01021 \\ 0.22526 & 0.97338 & -0.04238 & 0.00160 \\ -0.00663 & -0.04197 & -0.99910 & -0.00040 \\ 0.00959 & -0.00388 & -0.00031 & 0.99995 \end{pmatrix}, \quad (25)$$

and the corresponding absolute values for the deviations from the average experimental values (Eq. (20))

$$\delta V_{ud} = \begin{pmatrix} 0.091 & 0.117 & 2.339 \\ 0.431 & 1.418 & 1.348 \\ 2.951 & 0.358 & 1.559 \end{pmatrix}. \quad (26)$$

The corresponding total absolute average deviation, defined in Eq. (20), is 4.55785.

The diagonal values of the two mass matrices from Eq. (24) determine quark masses

$$\begin{aligned} \mathbf{M}_d^u / \text{MeV}/c^2 &= (1.3, 620.0, 172\,000., 700\,000.), \\ \mathbf{M}_d^d / \text{MeV}/c^2 &= (2.88508, 55.024, 2\,899.99, 700\,000.). \end{aligned} \quad (27)$$

2. We next choose $m_{u_4} = 1\,200$ GeV and $m_{d_4} = 1\,200$ Ge and fit the parameters of the two quark mass matrices (Eq. (1)) to the old experimental data [16]. We obtain the following two mass matrices

$$M^u = \begin{pmatrix} 351916. & 256894. & 257204. & 342714. \\ 256894. & 344411. & 342714. & 257204. \\ 257204. & 342714. & 341900. & 256894. \\ 342714. & 257204. & 256894. & 334395. \end{pmatrix}, M^d = \begin{pmatrix} 300783. & 299263. & 299288. & 300709. \\ 299263. & 300623. & 300709. & 299288. \\ 299288. & 300709. & 300856. & 299263. \\ 300709. & 299288. & 299263. & 300696. \end{pmatrix}, \quad (28)$$

and the mixing matrix

$$V_{ud} = \begin{pmatrix} -0.97425 & 0.22536 & -0.00301 & 0.00474 \\ 0.22534 & 0.97336 & -0.04239 & 0.00212 \\ -0.00663 & -0.04198 & -0.99910 & -0.00021 \\ 0.00414 & -0.00315 & -0.00011 & 0.99999 \end{pmatrix}. \quad (29)$$

The corresponding values for the deviations from the average experimental value of the matrix elements of the 3×3 submatrix are

$$\delta V_{ud} = \begin{pmatrix} 0.003 & 0.226 & 2.335 \\ 0.424 & 1.419 & 1.357 \\ 2.949 & 0.355 & 1.559 \end{pmatrix}. \quad (30)$$

The corresponding total average deviation (Eq. (20)) is now 4.55955.

The diagonal values of the two mass matrices from Eq. (28) determine quark masses

$$\begin{aligned} \mathbf{M}_d^u / \text{MeV}/c^2 &= (1.3, 620.0, 172\,000., 1\,200\,000.), \\ \mathbf{M}_d^d / \text{MeV}/c^2 &= (2.9, 55.0, 2\,900.0, 1\,200\,000.). \end{aligned} \quad (31)$$

One notices, that while the matrix elements of mass matrices of the u and the d quark change for a factor of ≈ 1.5 when changing the fourth family masses from 700 GeV to 1 200 GeV, becoming more "democratic" (that is the matrix elements become more and more equal), the mixing matrix elements of the 3×3 submatrix change very little (Eqs. (25, 29)). Let us add that the calculations, repeated with within the experimentally allowed intervals of twice three quark masses do not influence the results noticeably.

Let us now analyze the results obtained with the old data [16], Eq. (21), for the two choices of the fourth family masses: 700 GeV and 1 200 GeV. In both cases the rest two times three masses (Eqs. (27, 31)) are the same and in agreement with the experimental data (Eq. (23)). Variation of the three measured masses, in particular of the lowest 5 (m_u, m_c, m_d, m_s, m_b), within the experimentally declared intervals does not influence the mixing matrix noticeable. We shall demonstrate this with the new experimental results, when changing m_t for 3×760 MeV.

Let us, therefore, compare the two calculated mixing matrices, Eqs. (25, 29), with the old measured ones, Eq. (21). In Eq. (32) these three kinds of mixing matrix elements are denoted by (exp_o , old_1 and old_2) for the old experimental data, and the two calculated ones for the

choice for the fourth family masses: 700 GeV and 1 200 GeV, respectively.

$$|V_{(ud)_{old}}| = \begin{pmatrix} \begin{array}{c|ccc} exp_o & 0.97425 \pm 0.00022 & 0.2252 \pm 0.0009 & 0.00415 \pm 0.00049 \\ \hline old_1 & 0.97423 & 0.22531 & 0.00299 \\ old_2 & 0.97425 & 0.22536 & 0.00301 \\ \hline exp_o & 0.230 \pm 0.011 & 1.006 \pm 0.023 & 0.0409 \pm 0.0011 \\ \hline old_1 & 0.22526 & 0.97338 & 0.04238 \\ old_2 & 0.22534 & 0.97336 & 0.04239 \\ \hline exp_o & 0.0084 \pm 0.0006 & 0.0429 \pm 0.0026 & 0.89 \pm 0.07 \\ \hline old_1 & 0.00663 & 0.04197 & 0.99910 \\ old_2 & 0.00663 & 0.04198 & 0.99910 \end{array} \end{pmatrix}. \quad (32)$$

Comparing the two calculated mixing matrix elements (old_1 with $m_{u_4} = m_{d_4} = 700$ GeV and with $m_{u_4} = m_{d_4} = 1\,200$ GeV) with the measured ones, Eq. (21), we see:

a. The calculated matrix elements of the 3×3 submatrix of the 4×4 mixing matrix do not depend much on the masses of the fourth family members. As expected, they rise with the raising fourth family masses, but slightly, while the fourth family matrix elements, $V_{u_i d_4}$ and $V_{u_4 d_i}$, decrease, with the exception of one of them which even becomes larger, $V_{u_2 d_4}$. The accuracy with which we fitted the experimental data, (Eqs. (26, 26), smaller numbers mean better fitting, the number 2 means twice worse, while 0.1 means 10 times better fitting than it is the experimental inaccuracy) are not in average better (some are better, the others are worse).

b. Comparing the measured and the calculated matrix elements of the 3×3 submatrix we make prediction for next (now already known) measurements [15]:

b.i. The matrix element $V_{u_1 d_1}$ (V_{ud}) would very slightly decrease or stay unchanged, $V_{u_1 d_2}$ (V_{us}) will rise a little bit, and $V_{u_2 d_3}$ (V_{cb}) as well as $V_{u_3 d_3}$ (V_{tb}) will rise more.

b.ii. The matrix elements $V_{u_1 d_3}$ (V_{ub}), $V_{u_2 d_1}$ (V_{cd}), $V_{u_2 d_2}$ (V_{cs}), $V_{u_3 d_1}$ (V_{td}) and $V_{u_3 d_2}$ (V_{ts}) will lower.

Checking the new experimental values, Eq.(21), with these predictions one sees that the prediction is in agreement with the new experimental data, Eq. (22).

- **II.** Let us repeat the calculations with new experimental data [15] (Eqs. (23, 22)) to see how will the new data influence the mass matrices and the mixing matrix elements. Again we use the same two choices for the fourth family masses (they have quite different values and are enough illustrative, since any other two equally separated choices would lead to the

similar recognitions), $m_{u_4} = m_{d_4} = 700$ GeV and $m_{u_4} = m_{d_4} = 1\,200$ GeV.

Results of the calculations with the new experimental data (Eqs. (23, 22)) show smaller common deviations for the sums of all the average values of the nine matrix elements of the 3×3 submatrix (Eq. (20)) - 4.0715 and 4.0724 - than those with the old data - 4.5579 and 4.5596.

1. For $m_{u_4} = 700$ GeV and $m_{d_4} = 700$ GeV we get for the mass matrices

$$M^u = \begin{pmatrix} 226521. & 131887. & 132192. & 217715. \\ 131887. & 219347. & 217715. & 132192. \\ 132192. & 217715. & 216964. & 131887. \\ 217715. & 132192. & 131887. & 209790. \end{pmatrix}, M^d = \begin{pmatrix} 175776. & 174263. & 174288. & 175709. \\ 174263. & 175622. & 175709. & 174288. \\ 174288. & 175709. & 175857. & 174263. \\ 175709. & 174288. & 174263. & 175703. \end{pmatrix}, \quad (33)$$

for the mixing matrix

$$V_{ud} = \begin{pmatrix} -0.97423 & 0.22539 & -0.00299 & 0.00776 \\ 0.22534 & 0.97335 & -0.04245 & 0.00349 \\ -0.00667 & -0.04203 & -0.99909 & -0.00038 \\ 0.00677 & -0.00517 & -0.00020 & 0.99996 \end{pmatrix}. \quad (34)$$

The corresponding values (Eq. (20)) for the deviations from the average experimental values are

$$\delta V_{ud} = \begin{pmatrix} 0.074 & 0.109 & 2.336 \\ 0.043 & 0.791 & 1.042 \\ 2.891 & 0.753 & 0.685 \end{pmatrix}, \quad (35)$$

and the corresponding total absolute average deviation Eq. (20) is 4.07154.

The two mass matrices correspond to the diagonal masses

$$\begin{aligned} \mathbf{M}_d^u / \text{MeV}/c^2 &= (1.3, 620.0, 172\,000., 700\,000.), \\ \mathbf{M}_d^d / \text{MeV}/c^2 &= (2.9, 55.0, 2\,900.0, 700\,000.). \end{aligned} \quad (36)$$

2. For $m_{u_4} = 1\,200$ GeV and $m_{d_4} = 1\,200$ GeV we obtain the mass matrices

$$M^u = \begin{pmatrix} 354761. & 256877. & 257353. & 342539. \\ 256877. & 350107. & 342539. & 257353. \\ 257353. & 342539. & 336204. & 256877. \\ 342539. & 257353. & 256877. & 331550. \end{pmatrix}, M^d = \begin{pmatrix} 300835. & 299263. & 299288. & 300710. \\ 299263. & 300714. & 300710. & 299288. \\ 299288. & 300710. & 300765. & 299263. \\ 300710. & 299288. & 299263. & 300644. \end{pmatrix}, \quad (37)$$

and the mixing matrix

$$V_{ud} = \begin{pmatrix} 0.97423 & 0.22538 & 0.00299 & 0.00793 \\ -0.22514 & 0.97336 & 0.04248 & -0.00002 \\ 0.00667 & -0.04206 & 0.99909 & -0.00024 \\ -0.00773 & -0.00178 & 0.00022 & 0.99997 \end{pmatrix}. \quad (38)$$

The corresponding values for the deviations from the average experimental value for each matrix element are

$$\delta V_{ud} = \begin{pmatrix} 0.070 & 0.097 & 2.329 \\ 0.038 & 0.790 & 1.061 \\ 2.889 & 0.762 & 0.685 \end{pmatrix}. \quad (39)$$

The corresponding total average deviation Eq. (20) is 4.0724.

The two mass matrices correspond to the diagonal masses

$$\begin{aligned} \mathbf{M}_d^u / \text{MeV}/c^2 &= (1.3, 620.0, 172\,000., 1\,200\,000.), \\ \mathbf{M}_d^d / \text{MeV}/c^2 &= (2.88508, 55.024, 2\,899.99, 1\,200\,000.). \end{aligned} \quad (40)$$

- **III.** Let us check the sensitivity of our results with respect to changes of the measured quark masses within experimental inaccuracy. The influence of the changes of the lowest measured masses (m_u, m_d, m_s, m_c, m_b) within experimental inaccuracy is not noticeable, the only one which can change the results is the top mass. We therefore present the two calculated mixing matrices with $m_t = (172 \pm 3 \times 0.760)$ GeV fitted to the new experimental data [15].

1. First we do calculations for $m_{u_4} = m_{d_4} = 700$ GeV and $m_{u_3} = m_t = (172 + 3 \times 0.760)$

GeV with the new experimental data [15]. Below are the calculated mixing matrix

$$V_{ud} = \begin{pmatrix} -0.97424 & 0.22537 & -0.00299 & 0.00771 \\ 0.22533 & 0.97335 & -0.04246 & 0.00360 \\ -0.00666 & -0.04206 & -0.99909 & -0.00024 \\ -0.00670 & -0.00526 & -0.00020 & 0.99996 \end{pmatrix}, \quad (41)$$

and the corresponding values for the deviations from the average experimental value for each matrix element

$$\delta V_{ud} = \begin{pmatrix} 0.057 & 0.091 & 2.330 \\ 0.041 & 0.791 & 1.046 \\ 2.895 & 0.755 & 0.685 \end{pmatrix}. \quad (42)$$

The corresponding total average deviation Eq. (20) is 4.07154.

The two mass matrices correspond to the diagonal masses

$$\begin{aligned} \mathbf{M}_d^u/\text{MeV}/c^2 &= (1.3, 620.0, 174\,260., 700\,000.), \\ \mathbf{M}_d^d/\text{MeV}/c^2 &= (2.88508, 55.024, 2\,899.99, 700\,000.). \end{aligned} \quad (43)$$

2. We repeat the calculations for $m_{u_4} = m_{d_4} = 1\,200$ GeV and $m_{u_3} = m_t = (172 - 3 \times 0.760)$ GeV with the new experimental data [15]. Below are the calculated mixing matrix

$$V_{ud} = \begin{pmatrix} 0.97425 & 0.22542 & 0.00299 & 0.00466 \\ -0.22535 & 0.97335 & 0.04248 & -0.00216 \\ 0.00667 & -0.04205 & 0.99909 & -0.00021 \\ -0.00405 & -0.00526 & -0.00010 & 0.99999 \end{pmatrix}, \quad (44)$$

and the corresponding values for the deviations from the average experimental value for each matrix element

$$\delta V_{ud} = \begin{pmatrix} 0.017 & 0.146 & 2.336 \\ 0.044 & 0.791 & 1.058 \\ 2.883 & 0.761 & 0.685 \end{pmatrix}. \quad (45)$$

The corresponding total average deviation Eq. (20) is 4.07281.

The two mass matrices correspond to the diagonal masses

$$\begin{aligned}\mathbf{M}_d^u/\text{MeV}/c^2 &= (1.3, 620.0, 169\,740., 1\,200\,000.), \\ \mathbf{M}_d^d/\text{MeV}/c^2 &= (2.88508, 55.024, 2\,899.99, 1\,200\,000.). \end{aligned} \quad (46)$$

One notices, that in the case that the new experimental data [15] are used in the calculation, the matrix elements of mass matrices of the u -quarks differ less from those of the d -quarks than in the case when the old experimental data are used. The new experimental data lead to even more "democratic" mass matrices than the old ones, in particular is this the case for $m_{u_4} = m_{d_4} = 1\,200\text{ GeV}$.

The mixing matrix elements of the 3×3 submatrix calculated with the new data (those obtained with $m_{u_4} = m_{d_4} = 700\text{ GeV}$ are again close to those obtained with $m_{u_4} = m_{d_4} = 1\,200\text{ GeV}$ (Eqs. (34, 38))) agree better with the newer [15] than with the older [16] experimental values, as we predicted (results are presented in Eq. (32) and predictions made in **b.** below this equation) and already recognized.

Comparing the calculated mixing matrix elements obtained with $m_{u_4} = m_{d_4} = 700\text{ GeV}$ and either with $m_t = 172\text{ GeV}$ (Eq. (34)) or with $m_t = (172 + 3 \times 0.76)\text{ GeV}$ (Eq. (41)), as well as by comparing the calculated mixing matrix elements obtained with $m_{u_4} = m_{d_4} = 1\,200\text{ GeV}$ and with either $m_t = 172\text{ GeV}$ (Eq. (34)) or with $m_t = (172 - 3 \times 0.76)\text{ GeV}$ (Eq. (41)) one sees that the 3×3 submatrix changes in both cases very little, changes are almost negligible.

The matrix elements $V_{u_i d_4}$ and $V_{u_4 d_i}$ are, as expected, much stronger influenced.

We present in Eq. (47) the matrix elements of the 4×4 mixing matrix for quarks obtained when the 4×4 mass matrices respect the symmetry of Eq. (1) while we fit the parameters of the mass matrices to the old (exp_o) and the new (exp_n) experimental data. In both cases we present results for the choices of the fourth family quark masses: $m_{u_4} = m_{d_4} = 700\text{ GeV}$ (old_1, new_1) and $m_{u_4} = m_{d_4} = 1\,200\text{ GeV}$ (old_2, new_2). In parentheses, () and [], the changes of the matrix elements are presented, which are due to the changes of the top mass within the experimental inaccuracies: with the $m_t = (172 + 3 \times 0.76)\text{ GeV}$ and $m_t = (172 - 3 \times 0.76)$, respectively (if there is one number in parentheses only the last number is different, if there are two or more numbers in parentheses the last two or more numbers are different, if there is no parentheses no numbers

are different).

$$|V_{(ud)}| = \left(\begin{array}{c|cccc} \hline exp_o & 0.97425 \pm 0.00022 & 0.2252 \pm 0.0009 & 0.00415 \pm 0.00049 & \\ \hline exp_n & 0.97425 \pm 0.00022 & 0.2253 \pm 0.0008 & 0.00413 \pm 0.00049 & \\ \hline old_1 & 0.97423 & 0.22531 & 0.00299 & 0.01021 \\ old_2 & 0.97425 & 0.22536 & 0.00301 & 0.00474 \\ new_1 & 0.97423(4) & 0.22539(7) & 0.00299 & 0.00776(1) \\ new_2 & 0.97423[5] & 0.22538[42] & 0.00299 & 0.00793[466] \\ \hline exp_o & 0.230 \pm 0.011 & 1.006 \pm 0.023 & 0.0409 \pm 0.0011 & \\ \hline exp_n & 0.225 \pm 0.008 & 0.986 \pm 0.016 & 0.0411 \pm 0.0013 & \\ \hline old_1 & 0.22526 & 0.97338 & 0.04238 & 0.00160 \\ old_2 & 0.22534 & 0.97336 & 0.04239 & 0.00212 \\ new_1 & 0.22534(3) & 0.97335 & 0.04245(6) & 0.00349(60) \\ new_2 & 0.22531[5] & 0.97336[5] & 0.04248 & 0.00002[216] \\ \hline exp_o & 0.0084 \pm 0.0006 & 0.0429 \pm 0.0026 & 0.89 \pm 0.07 & \\ \hline exp_n & 0.0084 \pm 0.0006 & 0.0400 \pm 0.0027 & 1.021 \pm 0.032 & \\ \hline old_1 & 0.00663 & 0.04197 & 0.99910 & 0.00040 \\ old_2 & 0.00663 & 0.04198 & 0.99910 & 0.00021 \\ new_1 & 0.00667(6) & 0.04203(4) & 0.99909 & 0.00038 \\ new_2 & 0.00667 & 0.04206[5] & 0.99909 & 0.00024[21] \\ \hline old_1 & 0.00959 & 0.00388 & 0.00031 & 0.99995 \\ old_2 & 0.00414 & 0.00315 & 0.00011 & 0.99999 \\ new_1 & 0.00677(60) & 0.00517(26) & 0.00020 & 0.99996 \\ new_2 & 0.00773 & 0.00178 & 0.00022 & 0.99997[9] \\ \hline \end{array} \right). \quad (47)$$

Comparing the calculated mixing matrix elements for quarks, obtained by taking into account by the *spin-charge-family* theory suggested symmetry of mass matrices (Eq. (1)) with the measured ones, to which the calculated values are fitted, we notice:

A. The matrix elements of the 3×3 submatrix of the 4×4 mixing matrix depend very little on the masses of the fourth family members. As expected, they do rise with the raising fourth family masses, but very slightly, while the fourth family matrix elements, $V_{u_i d_4}$ and $V_{u_4 d_i}$, $i = (1, 2, 3)$, decrease (and correspondingly $V_{u_4 d_4}$ increase) in general, as expected, with the fourth family masses. But not all of them, the results with the new experimental data show that $V_{u_1 d_4}$, $V_{u_4 d_1}$ as well as $V_{u_4 d_3}$ even rise with the fourth family masses. The fourth family matrix elements are

sensitive also to the top quark mass. The accuracy with which we fitted the experimental data, [Eqs. (35, 39)] is better with the new experimental data and approximately the same for the two choices of the fourth family masses. The above (in item **b.**) predicted changes of mixing matrix elements were realized with the more accurate experimental data [15].

B. Changes of the lowest five quark masses within the experimental inaccuracy do not influence the results noticeably (and are correspondingly not presented among the calculated values). The presented results for the choice of the m_t mass within 3σ (Eqs. (41, 42, 44, 45)) show that 3×3 submatrix elements almost do not change within the experimental accuracy changed top mass. The elements $V_{u_i d_4}$ and $V_{u_4 d_i}$ are sensitive to the experimental accuracy of the top mass.

C. *Comparing the measured and the calculated matrix elements, of the 3×3 submatrix of the 4×4 mixing matrix we are making predictions for next more accurate measurements:*

C.i. The matrix element $V_{u_1 d_1}$ (V_{ud}) will stay the same or will very slightly decrease, $V_{u_1 d_2}$ (V_{us}) will still very slightly increase, $V_{u_2 d_1}$ (V_{cd}) will (after decreasing in exp_n [15]) very slightly increase towards exp_o , $V_{u_2 d_3}$ (V_{cb}) will still increase, and $V_{u_3 d_2}$ (V_{ts}) will (after decreasing in exp_n) slightly rise again (in the new experimental data [15], with worth accuracy, it is now too low).

C.ii. The matrix elements $V_{u_1 d_3}$ (V_{ub}) and $V_{u_2 d_2}$ (V_{cs}) will still lower, $V_{u_3 d_1}$ (V_{td}) should lower and $V_{u_3 d_3}$ (V_{tb}) must again lower.

C.iii. The fourth family masses change the mass matrices considerably, while their influence on the 3×3 submatrix of the 4×4 mixing matrix is quite weak. Accordingly there is very difficult to predict the fourth family masses from the today experimental data. We only can say, taking into account all the experimental evidences, that they might be around 1 TeV or above.

We can conclude: Requiring that the experimental data respect the symmetry of the mass matrices (Eq. (1)) (suggested by the *spin-charge-family* theory) the prediction can be made for the changes of the matrix elements of the 3×3 submatrix in future experiments. *The masses of the fourth family members are more difficult to be predicted, since they are very sensitive to the accuracy of the experimental data of the quark masses (only the top mass counts) and in particular to the accuracy of the experimental data of the quark mixing matrix, which is now too low. For the known fourth family masses the fourth family matrix elements of the mixing matrix can be predicted.* The fourth family matrix elements of the mixing matrix are even not very sensitive to the choice of the fourth family masses.

IV. DISCUSSIONS AND CONCLUSIONS

One of the most important open questions in the elementary particle physics is: Where do the families originate? Explaining the origin of families would answer the question about the number of families which are possibly observable at the low energy regime, about the origin of the scalar field(s) and Yukawa couplings and would also explain differences in the fermions properties - the differences in masses and mixing matrices among family members – quarks and leptons.

The *spin-charge-family* theory is offering a possible answer to the questions about the origin of families, of the scalar fields and the Yukawa couplings, as well as to several additional open questions of the *standard model* and also beyond it [14]. This theory predicts that there are four rather than so far observed three coupled families. The mass matrices of all the family members (quarks and leptons) demonstrate in the massless basis the $U(1) \times SU(2) \times SU(2)$ (each of the two $SU(2)$ is a subgroup, one of $SO(1,3)$ and the other of $SO(4)$) symmetry, Eq. (1).

Any accurate 3×3 submatrix of the 4×4 unitary matrix determines the 4×4 matrix uniquely. Since neither the quark and (in particular) nor the lepton 3×3 mixing matrix are measured accurately enough to be able to determine three complex phases of the 4×4 mixing matrix, we assume (what also simplifies the numerical procedure) that the mass matrices are symmetric and real and correspondingly the mixing matrices are orthogonal. We fitted the 6 free parameters of each family member mass matrix, Eq. (1), to twice three measured masses (6) of each pair of either quarks or leptons and to the 6 (from the experimental data extracted) parameters of the corresponding 4×4 mixing matrix.

We are presenting the results for quarks only. The accuracy of the experimental data for leptons are not yet accurate enough that would allow us meaningful predictions.

The numerical procedure, explained in this paper, to fit free parameters to the experimental data within the experimental inaccuracy of masses and in particular of the mixing matrices is very tough. It turned out that the experimental inaccuracies are too large to tell trustworthy mass intervals for the quarks masses of the fourth family members. Taking into account our calculations fitting the experimental data (and the meson decays evaluations in literature, as well as our own) we very roughly estimate that the fourth family quarks masses might be around 1 TeV or above.

Since the matrix elements of the 3×3 submatrix of the 4×4 mixing matrix depend very weakly on the fourth family masses, the calculated mixing matrix offer the prediction to what values will more accurate measurements move the present experimental data and also the fourth family mixing matrix elements in dependence of the fourth family masses, Eq. (47). The predictions are presented

in Subsect. IIIB, in particular in Eq. (47), and in comments below this equation.

We expect - detailed values can be found in Eq. (47) - that more accurate experiments will bring in comparison with the data [15]: a slightly smaller values for V_{ud} ; V_{ub} and V_{cs} will still lower, V_{td} will lower, V_{tb} will now lower; V_{us} will still slightly rise, V_{cd} will now slightly rise, V_{cb} will still rise and V_{ts} will now (after decreasing) increase.

We checked our ability to make predictions for the real experiment by first performing calculations with the old experimental data [16] and test the predictions on the new experimental data [15]. The results, presented in Eq. (32) and commented below Eq. (32) (b.), manifest that our predictions are in agreement with the new experimental data [15].

The fourth family mixing matrix elements depend, as expected, quite strongly on the fourth family masses. With the increasing fourth family masses they decrease, but not all (see Eq. (47)), $V_{u_1 d_4}$, $V_{u_4 d_1}$, and $V_{u_4 d_3}$ even rise with the rising four family masses. For chosen (quite large interval of the) masses of the fourth family members are their matrix elements quite accurately predicted (Eqs. (34, 38)).

Mass matrices are quite close to the "democratic" ones not only for leptons (which we not present in this paper) but also for quarks, Eqs. (33, 37). With the growing fourth family masses the "democracy" in matrix elements grow (Eqs. (33, 33)), as expected.

The complex mass matrices would lead to unitary and not to orthogonal mixing matrices. The more accurate experimental data for quarks would allow us to extract also the phases of the unitary mixing matrices, changing as well our predictions. In particular they will allow us to predict the fourth family masses much more accurately.

More accurately experimental data for leptons would allow us to make predictions also for leptons and to manifest that the *spin-charge-family* theory is explaining the properties of all the family members.

Appendix A: A brief presentation of the *spin-charge-family* theory

We present in this section a very brief introduction into the *spin-charge family* theory [1–14] to explain to readers the origin and properties of the families and correspondingly of the Higgs's scalar, with the weak and the hyper charge equal to $(\pm\frac{1}{2}, \mp\frac{1}{2})$, respectively, and the Yukawa couplings. This theory predicts the symmetry of the family members mass matrices, presented in Eq. (1). It also offers the explanation for the charges of the family members, for the appearance of the gauge fields to the family members charges, for the appearance of the dark matter and

for the matter-antimatter asymmetry. For better explanation we encourage the reader to read Refs. [13, 14].

There are, namely, two (only two) kinds of the anticommuting Clifford algebra objects: The Dirac γ^a take care of the spin in $d = (3 + 1)$, the spin in $d \geq 4$ (rather than the total angular momentum) manifests in $d = (3 + 1)$ in the low energy regime as the charges. In this part the *spin-charge family* theory is like the Kaluza-Klein theory [45, 46], unifying spin and charges, and offering a possible answer to the question about the origin of the so far observed charges. It also explains why left handed family members are weak charged, while the right handed are weak chargeless.

The second kind of the Clifford algebra objects, forming the equivalent representations with respect to the Dirac kind, recognized by one of the authors (SNMB), is responsible for the appearance of families of fermions.

There are correspondingly also two kinds of gauge fields, which manifest in $d = (3 + 1)$ as the vector gauge fields and as the scalar gauge fields. The vector gauge fields explain properties of all known vector gauge fields. The scalar gauge fields with the space index s , $5 \leq s \leq 8$, offer, as weak doublets carrying also appropriate hyper charge, explanations for the appearance of the Higgs scalar and the Yukawa coupling. Those scalars with the space index s , $9 \leq s \leq 14$, are colour triplets causing transitions of antileptons into quarks and antiquarks into quarks and back. In the presence of the scalar condensate, which breaks the matter-antimatter symmetry, offer these scalars the explanation for the observed matter-antimatter asymmetry, explaining also the proton decay. All the scalar fields carry besides the charges, determined by the space index, also additional charges, which all are in the adjoint representations.

In the *spin-charge-family* theory originate all the properties of at low energies observed fermions and bosons in a simple starting action for massless fields in $d = [1 + (d - 1)]$. The theory makes a choice of $d = (1 + 13)$, since one Weyl representation of $SO(1, 13)$ contains all the family members, left and right handed, with antimembers included.

Fermions interact (Eq. (A1)) with the vielbeins f^α_a and also with the two kinds of the spin connection fields: with $\omega_{abc} = f^\alpha_c \omega_{ab\alpha}$, which are the gauge fields of $S^{ab} = \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a)$, and with $\tilde{\omega}_{abc} = f^\alpha_c \tilde{\omega}_{ab\alpha}$, which are the gauge fields of $\tilde{S}^{\tilde{a}\tilde{b}} = \frac{i}{4}(\tilde{\gamma}^{\tilde{a}} \tilde{\gamma}^{\tilde{b}} - \tilde{\gamma}^{\tilde{b}} \tilde{\gamma}^{\tilde{a}})$. α, β, \dots is the Einstein

index and a, b, \dots is the flat index. The starting action is the simplest one

$$\begin{aligned}
S &= \int d^d x E \mathcal{L}_f + \int d^d x E (\alpha R + \tilde{\alpha} \tilde{R}), \\
\mathcal{L}_f &= \frac{1}{2} (\bar{\psi} \gamma^a p_{0a} \psi) + h.c. \\
p_{0a} &= f^\alpha{}_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^\alpha{}_a\}_-, \quad p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{\tilde{a}\tilde{b}} \tilde{\omega}_{\tilde{a}\tilde{b}\alpha}, \\
R &= \frac{1}{2} \{f^{\alpha[a} f^{\beta b]} (\omega_{ab\alpha, \beta} - \omega_{ca\alpha} \omega^c{}_{b\beta})\} + h.c., \quad \tilde{R} = \frac{1}{2} f^{\alpha[\tilde{a}} f^{\beta \tilde{b}]} (\tilde{\omega}_{\tilde{a}\tilde{b}\alpha, \beta} - \tilde{\omega}_{\tilde{c}\tilde{a}\alpha} \tilde{\omega}^{\tilde{c}}{}_{\tilde{b}\beta}) + h.c..
\end{aligned} \tag{A1}$$

$$\tag{A2}$$

$E = \det(e^a_\alpha)$ and $e^a_\alpha f^\beta{}_a = \delta^\beta_\alpha$. The vielbeins $f^\alpha{}_{\tilde{a}} = f^\alpha{}_a$ ([13]).

Fermions, coupled to the vielbeins and the two kinds of the spin connection fields, *manifest* (after several breaks and the appearance of the scalar condensate of the two right handed neutrinos [7, 13, 14, 47]) *before the electroweak break four coupled massless families of quarks and leptons*, the left handed fermions are weak charged and the right handed ones are weak chargeless, explaining all the assumptions of the *standard model*.

The vielbeins and the two kinds of the spin connection fields manifest effectively as the observed gauge fields and (those with the scalar indices with respect to $d = (1 + 3)$) as several scalar fields. The mass matrices of the four family members (quarks and leptons) are after the electroweak break expressible on a tree level by the vacuum expectation values of the two kinds of the spin connection fields and the corresponding vielbeins with the scalar indices $s = (7, 8)$ ([1, 2, 6, 7, 12, 13]):

i. One kind, originating in the scalar fields $\tilde{\omega}_{abc}$, manifests as the two $SU(2)$ triplets – $\tilde{A}_s^{\tilde{N}_L i}, i = (1, 2, 3), s = (7, 8)$; $\tilde{A}_s^{\tilde{1} i}, i = (1, 2, 3), s = (7, 8)$; – and one singlet – $\tilde{A}_s^{\tilde{4}}, s = (7, 8)$ – contributing equally to all the family members.

ii. The second kind originates in the scalar fields ω_{abc} , manifesting as the three singlets – $A_s^Q, A_s^{Q'}, A^{Y'}$, $s = (7, 8)$ (Q is the electromagnetic charge, Q' is the non conserved charge of the Z boson, Y' originates in the second $SU(2)$, which breaks at the appearance of the condensate, leaving the hyper charge Y conserved) – contributing the same values to all the families and distinguishing among family members.

The scalar fields with $s = (7, 8)$ "dress" the right handed quarks and leptons with the hyper charge and the weak charge so that they manifest the charges of the left handed partners, explaining assumptions for the Higgs's scalar of the *standard model* [51] and contributing also to the masses of the weak bosons, as doublets with respect to the weak charge.

Loop corrections, to which all the scalar and also gauge vector fields contribute coherently,

change contributions of the off-diagonal and diagonal elements appearing on the tree level, keeping the tree level symmetry of mass matrices unchanged [52].

1. Symmetries of the mass matrices on the tree level and beyond manifesting the

$SU(2) \times SU(2) \times U(1)$ symmetry

Let us make a choice of a massless basis ψ_i^α , $i = (1, 2, 3, 4)$, for a particular family member $\alpha = (u, \nu, d, e)_{L,R}$. The two $SU(2)$ operators [53], \tilde{N}_L^i and $\tilde{\tau}^{1i}$,

$$\begin{aligned} \tilde{N}_L^i, i = (1, 2, 3), \quad \tau_L^i, i = (1, 2, 3), \\ \{\tilde{N}_L^i, \tilde{N}_L^j\}_- = i \varepsilon^{ijk} \tilde{N}_L^k, \quad \{\tilde{\tau}^{1i}, \tilde{\tau}^{1j}\}_- = i \varepsilon^{ijk} \tilde{\tau}^{1k}, \quad \{\tilde{N}_L^i, \tilde{\tau}^{1j}\}_- = 0, \end{aligned} \quad (\text{A3})$$

ε^{ijk} is the totally antisymmetric tensor, transform the basic vectors ψ_i^α , into one another as follows

$$\begin{aligned} \tilde{N}_L^3 (\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha) &= \frac{1}{2} (-\psi_1^\alpha, \psi_2^\alpha, -\psi_3^\alpha, \psi_4^\alpha), \\ \tilde{N}_L^+ (\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha) &= (\psi_2^\alpha, 0, \psi_4^\alpha, 0), \\ \tilde{N}_L^- (\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha) &= (0, \psi_1^\alpha, 0, \psi_3^\alpha), \\ \tilde{\tau}^{13} (\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha) &= \frac{1}{2} (-\psi_1^\alpha, -\psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha), \\ \tilde{\tau}^{1+} (\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha) &= (\psi_3^\alpha, \psi_4^\alpha, 0, 0), \\ \tilde{\tau}^{1-} (\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha) &= (0, 0, \psi_1^\alpha, \psi_2^\alpha). \end{aligned} \quad (\text{A4})$$

The three $U(1)$ operators (Q, Q' and Y') commute with the family operators \tilde{N}_L^i and $\tilde{\tau}^{1i}$, distinguishing only among family members α

$$\begin{aligned} \{\tilde{N}_L^i, (Q, Q', Y')\}_- &= (0, 0, 0), \\ \{\tilde{\tau}^{1i}, (Q, Q', Y')\}_- &= (0, 0, 0), \\ (Q, Q', Y') (\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha) &= (Q^\alpha, Q'^\alpha, Y'^\alpha) (\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha), \end{aligned} \quad (\text{A5})$$

giving the same eigenvalues for all the families.

The nonzero vacuum expectation values of the gauge scalar fields of \tilde{N}_L^i ($\tilde{A}_s^{\tilde{N}_L^i} = \tilde{C}^{\tilde{N}_L^i}{}_{\tilde{m}\tilde{n}} \tilde{\omega}^{\tilde{m}\tilde{n}}_s$, $(\tilde{m}, \tilde{n}) = (0, 1, 2, 3)$), of $\tilde{\tau}^{1i}$ ($\tilde{A}_s^{\tilde{\tau}^{1i}} = \tilde{C}^{\tilde{\tau}^{1i}}{}_{\tilde{t}\tilde{t}'} \tilde{\omega}^{\tilde{t}\tilde{t}'}_s$, $(\tilde{t}, \tilde{t}') = (5, 6, 7, 8)$) and of the three singlet gauge scalar fields of (Q, Q' and Y'), which all are superposition of $\omega_{t,t',s}$ ($A_s^Q = C^Q{}_{tt'} \omega^{tt'}_s$, $A_s^{Q'} = C^{Q'}{}_{tt'} \omega^{tt'}_s$ and $A_s^{Y'} = C^{Y'}{}_{tt'} \omega^{tt'}_s$, $(s, t, t') = (5, 6, 7, 8)$), determine on the tree level, together with the corresponding coupling constants, the $SU(2) \times SU(2) \times U(1)$ symmetry and the strength of the

mass matrix of each family member α , Eq. (1). In loop corrections all the scalar fields - $\tilde{A}_s^{\tilde{N}_L i}$, $\tilde{A}_s^{\tilde{1} i}$, $A_s^Q, A_s^{Q'}, A_s^{Y'}$ - contribute to all the matrix elements, keeping the symmetry unchanged, Eq. (1). The twice two zeros on the tree level obtain in loop corrections the value b .

One easily checks that a change of the phases of the left and the right handed members, there are $(2n - 1)$ possibilities, causes changes in phases of matrix elements in Eq. (1).

All the scalars are doublets with respect to the weak charge, contributing to the weak and the hyper charge of the fermions so that they transform the right handed members into the left handed ones [13], what is in the *standard model* just required.

Appendix B: Properties of non Hermitian mass matrices

This pedagogic presentation of well known properties of non Hermitian matrices can be found in many textbooks, for example [44]. We repeat this topic here only to make our discussions transparent.

Let us take a non hermitian mass matrix M^α as it follows from the *spin-charge-family* theory, α denotes a family member (index \pm used in the main text is dropped).

We always can diagonalize a non Hermitian M^α with two unitary matrices, S^α ($S^{\alpha\dagger} S^\alpha = I$) and T^α ($T^{\alpha\dagger} T^\alpha = I$)

$$S^{\alpha\dagger} M^\alpha T^\alpha = \mathbf{M}_d^\alpha = (m_1^\alpha \dots m_i^\alpha \dots m_n^\alpha). \quad (\text{B1})$$

The proof is added below.

Changing phases of the basic states, those of the left handed one and those of the right handed one, the new unitary matrices $S'^\alpha = S^\alpha F_{\alpha S}$ and $T'^\alpha = T^\alpha F_{\alpha T}$ change the phase of the elements of diagonalized mass matrices \mathbf{M}_d^α

$$\begin{aligned} S'^{\alpha\dagger} M^\alpha T'^\alpha &= F_{\alpha S}^\dagger \mathbf{M}_d^\alpha F_{\alpha T} = \\ &= \text{diag}(m_1^\alpha e^{i(\phi_1^{\alpha S} - \phi_1^{\alpha T})} \dots m_i^\alpha e^{i(\phi_i^{\alpha S} - \phi_i^{\alpha T})} \dots m_n^\alpha e^{i(\phi_n^{\alpha S} - \phi_n^{\alpha T})}), \\ F_{\alpha S} &= \text{diag}(e^{-i\phi_1^{\alpha S}}, \dots, e^{-i\phi_i^{\alpha S}}, \dots, e^{-i\phi_n^{\alpha S}}), \\ F_{\alpha T} &= \text{diag}(e^{-i\phi_1^{\alpha T}}, \dots, e^{-i\phi_i^{\alpha T}}, \dots, e^{-i\phi_n^{\alpha T}}). \end{aligned} \quad (\text{B2})$$

In the case that the mass matrix is Hermitian T^α can be replaced by S^α , but only up to phases originating in the phases of the two basis, the left handed one and the right handed one, since they remain independent.

One can diagonalize the non Hermitian mass matrices in two ways, that is either one diagonalizes $M^\alpha M^{\alpha\dagger}$ or $M^{\alpha\dagger} M^\alpha$

$$\begin{aligned} (S^{\alpha\dagger} M^\alpha T^\alpha)(S^{\alpha\dagger} M^\alpha T^\alpha)^\dagger &= S^{\alpha\dagger} M^\alpha M^{\alpha\dagger} S^\alpha = \mathbf{M}_{dS}^{\alpha 2}, \\ (S^{\alpha\dagger} M^\alpha T^\alpha)^\dagger (S^{\alpha\dagger} M^\alpha T^\alpha) &= T^{\alpha\dagger} M^{\alpha\dagger} M^\alpha T^\alpha = \mathbf{M}_{dT}^{\alpha 2}, \\ \mathbf{M}_{dS}^{\alpha\dagger} &= \mathbf{M}_{dS}^\alpha, \quad \mathbf{M}_{dT}^{\alpha\dagger} = \mathbf{M}_{dT}^\alpha. \end{aligned} \quad (\text{B3})$$

One can prove that $\mathbf{M}_{dS}^\alpha = \mathbf{M}_{dT}^\alpha$. The proof proceeds as follows. Let us define two Hermitian (H_S^α, H_T^α) and two unitary matrices (U_S^α, U_T^α)

$$\begin{aligned} H_S^\alpha &= S^\alpha \mathbf{M}_{dS}^\alpha S^{\alpha\dagger}, & H_T^\alpha &= T^\alpha \mathbf{M}_{dT}^{\alpha\dagger} T^{\alpha\dagger}, \\ U_S^\alpha &= H_S^{\alpha-1} M^\alpha, & U_T^\alpha &= H_T^{\alpha-1} M^{\alpha\dagger}. \end{aligned} \quad (\text{B4})$$

It is easy to show that $H_S^{\alpha\dagger} = H_S^\alpha$, $H_T^{\alpha\dagger} = H_T^\alpha$, $U_S^\alpha U_S^{\alpha\dagger} = I$ and $U_T^\alpha U_T^{\alpha\dagger} = I$. Then it follows

$$\begin{aligned} S^{\alpha\dagger} H_S^\alpha S^\alpha &= \mathbf{M}_{dS}^\alpha = \mathbf{M}_{dS}^{\alpha\dagger} = S^{\alpha\dagger} M^\alpha U_S^{\alpha-1} S^\alpha = S^{\alpha\dagger} M^\alpha T^\alpha, \\ T^{\alpha\dagger} H_T^\alpha T^\alpha &= \mathbf{M}_{dT}^\alpha = \mathbf{M}_{dT}^{\alpha\dagger} = T^{\alpha\dagger} M^{\alpha\dagger} U_T^{\alpha-1} T^\alpha = T^{\alpha\dagger} M^{\alpha\dagger} S^\alpha, \end{aligned} \quad (\text{B5})$$

where we recognized $U_S^{\alpha-1} S^\alpha = T^\alpha$ and $U_T^{\alpha-1} T^\alpha = S^\alpha$. Taking into account Eq. (B2) the starting basis can be chosen so, that all diagonal masses are real and positive.

Acknowledgments

The author acknowledges funding of the Slovenian Research Agency.

-
- [1] N.S. Mankoč Borštnik, "Spin-charge-family theory is explaining appearance of families of quarks and leptons, of Higgs and Yukawa couplings", in *Proceedings to the 16th Workshop "What comes beyond the standard models"*, Bled, 14-21 of July, 2013, eds. N.S. Mankoč Borštnik, H.B. Nielsen and D. Lukman (DMFA Založništvo, Ljubljana, December 2013) p.113 [arXiv:1312.1542].
 - [2] N.S. Mankoč Borštnik, "Do we have the explanation for the Higgs and Yukawa couplings of the *standard model*", <http://arxiv.org/abs/1212.3184v2>, (<http://arxiv.org/abs/1207.6233>), in *Proceedings to the 15 th Workshop "What comes beyond the standard models"*, Bled, 9-19 of July, 2012, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana, December 2012, p.56-71, [arxiv.1302.4305].
 - [3] N.S. Mankoč Borštnik, *Phys. Lett. B* **292**, 25 (1992).
 - [4] N.S. Mankoč Borštnik, *J. Math. Phys.* **34**, 3731 (1993).

- [5] N.S. Mankoč Borštnik, *Int. J. Theor. Phys.* **40**, 315 (2001).
- [6] A. Borštnik Bračič and N.S. Mankoč Borštnik, *Phys. Rev. D* **74**, 073013 (2006) [hep-ph/0301029; hep-ph/9905357, p. 52-57; hep-ph/0512062, p.17-31; hep-ph/0401043 ,p. 31-57].
- [7] N.S. Mankoč Borštnik, *J. of Modern Phys.* **4**, 823 (2013) [arXiv:1312.1542].
- [8] N.S. Mankoč Borštnik, *Modern Phys. Lett. A* **10**, 587 (1995).
- [9] G. Bregar, M. Breskvar, D. Lukman and N.S. Mankoč Borštnik, *New J. of Phys.* **10**, 093002 (2008), [arXiv:0606159, aeXiv:07082846, arXiv:0612250, p.25-50].
- [10] G. Bregar and N.S. Mankoč Borštnik, *Phys. Rev. D* **80**, 083534 (2009).
- [11] G. Bregar, N.S. Mankoč Borštnik, "Can we predict the fourth family masses for quarks and leptons?", Proceedings to the 16 th Workshop "What comes beyond the standard models", Bled, 14-21 of July, 2013, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2013, p. 31-51, [arXiv:1403.4441].
- [12] N.S. Mankoč Borštnik, "Do we have the explanation for the Higgs and Yukawa couplings of the *standard model*", [arXiv:1212.3184, arXiv:1011.5765].
- [13] N.S. Mankoč Borštnik, "The *spin-charge-family* theory explains why the scalar Higgs carries the weak charge $\pm\frac{1}{2}$ and the hyper charge $\mp\frac{1}{2}$ ", Proceedings to the 17th Workshop "What Comes Beyond the Standard Models", Bled, July 20 - 28, 2014 [arXiv:1409.7791, arXiv:1212.4055].
- [14] N.S. Mankoč Borštnik, *Phys. Rev. D* **91** (2015) 6, 065004 ID: 0703013. doi:10.1103; [arXiv:1409.7791, arXiv:1502.06786v1].
- [15] A.Ceccucci (CERN), Z.Ligeti (LBNL), Y. Sakai (KEK), Particle Data Group, Aug. 29, 2014, [http://pdg.lbl.gov/2014/reviews/rpp2014-rev-ckm-matrix.pdf].
- [16] K. Nakamura *et al.*, (Particle Data Group), *J. Phys. G*: **37** 075021 (2010); Z.Z. Xing, H. Zhang, S. Zhou, *Phys. Rev. D* **77** (2008) 113016; Beringer et al, *Phys. Rev. D* **86** (2012) 010001, Particle Physics booklet, July 2012, PDG, APS physics.
- [17] H. Fritzsch, *Phys. Lett.* **73 B**, 317 (1978); *Nucl. Phys.* **B 155** (1979) 189, *Phys. Lett.* **B 184** (1987) 391.
- [18] C.D. Froggatt, H.B. Nielsen, *Nucl. Phys.* **B 147** (1979) 277.
- [19] C. Jarlskog, *Phys. Rev. Lett.* **55** (1985) 1039.
- [20] G.C. Branco and D.-D. Wu, *ibid.* **205** (1988) 253.
- [21] H. Harari, Y. Nir, *Phys. Lett.* **B 195** (1987) 586.
- [22] E.A. Paschos, U. Turke, *Phys. Rep.* **178** (1989) 173.
- [23] C.H. Albright, *Phys. Lett.* **B 246** (1990) 451.
- [24] Zhi-Zhong Xing, *Phys. Rev. D* **48** (1993) 2349.
- [25] D.-D. Wu, *Phys. Rev. D* **33** (1996) 860.
- [26] E.J. Chun, A. Lukas, [arxiv:9605377v2].
- [27] B. Stech, *Phys. Lett.* **B 403** (1997) 114.
- [28] E. Takasugi, M. Yashimura, [arxiv:9709367].

- [29] G. Altarelli, NJP 6 (2004) 106.
- [30] S. Tatur, J. Bartelski, Phys. Rev. **D74** (2006) 013007, [arXiv:0801.0095v3].
- [31] A. Kleppe, [arXiv:1301.3812].
- [32] P. O. Ludl and W. Grimus, JHEP 1407 (2014) 090 [arXiv:1406.3546, arXiv:1501.04942].
- [33] J. Erler, P. Langacker, arXiv:1003.3211.
- [34] W.S. Hou, C.L. Ma, arXiv:1004.2186.
- [35] Yu.A. Simonov, [arXiv:1004.2672].
- [36] A.N. Rozanov, M.I. Vysotsky, [arXiv:1012.1483].
- [37] CBC News, Mar 15, 2013 9:05.
- [38] C. Jarlskog, [arxiv:math-ph/0504049]
- [39] K. Fujii, [arXiv:math-ph/0505047v3].
- [40] S. Rosati, INFN Roma, talk at Miami 2012, Atlas collaboration.
- [41] D. Lukman, N.S. Mankoč Borštnik, "Families of spinors in $d = (1 + 5)$ with zweibein and two kinds of spin connection fields on an almost S^2 ", [arXiv:1212.2370].
- [42] A. Hernandez-Galeana, N.S. Mankoč Borštnik, "Masses and Mixing matrices of families of quarks and leptons within the Spin-Charge-Family theory, Predictions beyond the tree level", [arXiv:1112.4368 p. 105-130, arXiv:1012.0224 p. 166-176].
- [43] M.I. Vysotsky, arXiv:1312.0474; A. Lenz, Adv. High Energy Phys. **2013** (2013) 910275.
- [44] Ta-Pei Cheng, Ling-Fong Li, *Gauge theory of elementary particles*, Clarendon Press Oxford, 1984.
- [45] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys. **96** (1921) 69, O. Klein, Z.Phys. **37** (1926) 895.
- [46] The authors of the works presented in *An introduction to Kaluza-Klein theories*, Ed. by H. C. Lee, World Scientific, Singapore 1983, T. Appelquist, A. Chodos, P.G.O. Freund (Eds.), *Modern Kaluza-Klein Theories*, Reading, USA: Addison Wesley, 1987.
- [47] D. Lukman, N. S. Mankoč Borštnik, H. B. Nielsen, *New J. Phys.* **13** (2011) 10302 [arXiv:1001.4679v4].
- [48] M.I. Vysotsky and A. Lenz comment in their papers [43] that the fourth family is excluded provided that one assumes the validity of the *standard model* with one scalar field (the scalar Higgs) while extending the number of families from three to four when, in loop corrections, evaluating the decay properties of the scalar Higgs. We have, however, several scalars: Two times three triplets with respect to the family quantum numbers and three singlets, which distinguish among the family members [13], all these scalars carry the weak and the hyper charge as the scalar Higgs. These scalar fields determine all the masses and the mixing matrices of quarks and leptons and of the weak gauge fields, what in the *standard model* is achieved by the choice of the scalar Higgs properties and the Yukawa couplings. Our rough estimations of the decay properties of mesons show that the fourth family quarks might have masses close to 1 TeV or above.
- [49] In Ref. [9] we made a similar assumption, except that we allow there that the symmetry of mass matrices, manifesting on the tree level, might be changed in loop corrections. We got in that study

- dependence of mass matrices and correspondingly mixing matrices on masses of the fourth family members. The study of the symmetry of mass matrices in loop corrections in the massless basis shows that loop corrections keep the symmetry, determined by the group content $SU(2) \times SU(2) \times U(1)$ in all orders [42].
- [50] There are also Majorana like terms contributing in higher order loop corrections [7], which might strongly influence in particular the neutrino mass matrix.
- [51] It is the term $\gamma^0 \gamma^s \phi_s^{Ai}$, where ϕ_s^{Ai} , with $s = (7, 8)$, denotes any of the scalar fields, which transforms the right handed fermions into the corresponding left handed partner [1, 2, 6, 7, 13? , 14]. This mass term originates in $\bar{\psi} \gamma^a p_{0a} \psi$ of the action Eq.(A1), with $a = s = (7, 8)$ and $p_{0s} = f_s^\sigma (p_\sigma - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{\tilde{a}\tilde{b}\sigma} - \frac{1}{2} S^{st} \omega_{st\sigma})$.
- [52] It can be seen that all the loop corrections keep the starting symmetry of the mass matrices unchanged in the massless basis. We have also started [7, 42] with the evaluation of the loop corrections to the tree level values. This estimation has been done so far [42] only up to the first order and partly to the second order.
- [53] The infinitesimal generators \tilde{N}_L^i , $i = (1, 2, 3)$ and \tilde{N}_R^i , $i = (1, 2, 3)$ determine the algebra of the two invariant subgroups of the $\widetilde{SO}(1, 3)$ group, while $\tilde{\tau}^{1i}$, $i = (1, 2, 3)$ and $\tilde{\tau}^{2i}$, $i = (1, 2, 3)$ determine the two invariant subgroups of the $\widetilde{SO}(4)$ group. The four families, discussed in this paper, carry family quantum numbers of \tilde{N}_L^i and $\tilde{\tau}^{1i}$.